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A Summability Approximation Theorem for the Transforms of the Geometric Series

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The behaviour of summability transforms of power series outside their circle of convergence has been studied by various authors. In particular, the question of where and to what limits a regular method can sum the geometric series outside the unit disc has been the object of many investigations. A result by W. Luh [*Mitt. Math. Sem. Giessen* **113** (1974)] suggests that the limits and the sets of convergence may be chosen from surprisingly large classes. However, Luh's matrices are not regular. In this paper, it is shown that the construction method used in Luh does yield regular matrices if the main construction tool, Mergelyan's theorem, is replaced by an approximation theorem (proved in Section 2) which appears to be more appropriate for the needs of summability theory. Two of many possible applications of this theorem are given in Section 3, the second of which improves upon the main result in Luh.

1. INTRODUCTION

Many authors have investigated the effectiveness of summability methods on power series outside their circle of convergence. See, for example, [7, 8, 10, 12, 13, 16–20]. To a certain degree, the Okada theorem (a general version of which can be found in [2]) allows us to extend results about the geometric series to general power series. In this paper, we present an approximation theorem which enables us to give a satisfactory solution to the problem of constructing a regular matrix that sums the geometric series to prescribed values on a given set S.

The existence of such methods was proved by Israpilov [4–6] for the case that S is a countable set outside the unit circle. However, a result by Luh (Theorem 5.1 in [9]) suggests that S and the prescribed limits of the transforms may be chosen from a much larger class. In fact, the matrices constructed in [9] sum the geometric series on a countable collection C of points, Jordan arcs, and simply connected regions.

Unfortunately, the methods A obtained in [9] are not regular in general.

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This weakness will be eliminated with the help of the approximation theorem proved in Section 2. It will be shown that if C does not intersect the closed unit disc then A can be chosen from the following class.

DEFINITION. Let $\mathscr{R}$ be the set of all row-finite matrices $(a_{n,k})_{n,k=0}^{\infty}$ which satisfy the conditions

(i) $\lim_{n\to\infty} \max_{k\geq 0} |a_{n,k}| = 0$,

(ii) $\sum_{k=0}^{\infty} a_{n,k} = 1$ for $n = 0, 1, \ldots$,

(iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{n,k}| = 1$.

It is easily seen that (i), (ii), (iii) imply the three Toeplitz conditions; therefore all matrices in $\mathscr{R}$ are regular. Moreover, it can be shown that $\mathscr{R}$ is a semigroup under the usual multiplication of summability methods.

Further notations. For every set $S \subset \mathbb{C}$, let $\mathring{S}$ denote the interior and $\bar{S}$ the closure of S. A sequence of functions (f_n) will be called compactly convergent to a function f on $S \subset \mathbb{C}$ if it converges to f uniformly on every compact subset of S. Also, the following abbreviations will be used throughout:

$$\mathbb{D}_r = \{z \in \mathbb{C} : |z| \leqslant r\} \quad \text{for} \quad r \geqslant 0,\ \mathbb{D} = \mathbb{D}_1,\ \mathbb{D}_\infty = \mathbb{C},$$

$$U_\delta(z) = \{t \in \mathbb{C} : |t - z| < \delta\} \quad \text{for} \quad \delta > 0,\ z \in \mathbb{C},$$

$$U_\delta(S) = \{t \in \mathbb{C} : |t - w| < \delta \text{ for some } w \in S\} \quad \text{for} \quad \delta > 0,\ S \subset \mathbb{C}.$$

If K is a compact set then $A(K)$ denotes the Banach space of all functions which are continuous on K and holomorphic on $\mathring{K}$. Finally, we make the

DEFINITION. For every compact set K with $0 \in \mathring{K}$, $1 \notin K$, let

$$M(K) = \{z \in \mathbb{C} : 1 < |z| < 1 + \delta,\ \operatorname{dist}(z, K^c) < \delta\} \cup U_\delta(1),$$

where $1 - \delta/2 = \operatorname{dist}(0, K^c)$.

2. A SUMMABILITY APPROXIMATION THEOREM

Suppose that we want to construct a row-finite summability method $A = (a_{n,k})$ with certain properties. Let us denote the A-transform of the geometric sequence (z^n) by (τ_n), i.e.,

$$\tau_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k.$$

Also, denote the A-transform of the geometric series by (σ_n), i.e.,

$$\sigma_n(z) = \sum_{k=0}^{\infty} a_{n,k}(1 + \cdots + z^k).$$

The Mergelyan approximation theorem (see, for example, [11]) asserts that if K_n is a compact set which does not separate the plane and if $g \in A(K_n)$, then there is a polynomial p_n such that $|p_n - g| < 1/n$ on K_n. Therefore, if we wish the transform (τ_n) to converge to g on a given set S, we can proceed in the following manner: we choose (K_n) to be a sequence which exhausts S and determine the entries $a_{n,k}$ by setting $\tau_n = p_n$. Since $\sigma_n(z) = (1/(1-z))(\tau_n(1) - z\tau_n(z))$ for $z \neq 1$, the result "$\sigma_n \to f$ on S" follows if $g(1) = 1$ and $g(z) = (1/z)(1 - (1-z)f(z))$ for $z \in S$.

For example, suppose that $S = \mathring{D} \cup \{e^{i\theta}: \pi/2 \leqslant \theta < \pi\}$ and that f and g are defined by $f(z) = 1/(1-z)$ for $|z| < 1$, $f(z) = 1/(1+z)$ for $z \in S \backslash \mathring{D}$ and $g(1) = 1$, $g(z) = 0$ for $|z| < 1$, $g(z) = 2/(1+z)$ for $z \in S \backslash \mathring{D}$. Set $K_n = \{1\} \cup D_{n/(n+1)} \cup \{e^{i\theta}: \pi/2 \leqslant \theta \leqslant \pi n/(n+1)\}$ and determine τ_n and σ_n $(n \geqslant 1)$ as in the preceding paragraph. It follows that $\tau_n \to g$ and $\sigma_n \to f$ on S.

In further applications the Mergelyan theorem and the construction method described above can be used to prove a variety of existence theorems in summability theory (cf. [8–10]). The method works for all sets S which can be exhausted by a sequence of compact sets K_n that do not separate the plane. If $g \in A(K_n)$ for all K_n then we obtain a row-finite matrix A such that $\tau_n \to g$ on S. This matrix satisfies the row-sum condition if $g(1) = 1$; and it is not hard to show that A also satisfies the column-limit condition if $g(z) = 0$ in a neighbourhood of 0. However, the matrices obtained are not necessarily regular. Indeed, in our example g is not bounded on S and so (τ_n) is not uniformly bounded on $|z| = 1$ whence A cannot satisfy the row-norm condition.

The proposed construction method does provide regular matrices if the following adjustment is made: we replace the Mergelyan theorem by

THEOREM 1. *Suppose K and L are compact sets not separating the plane such that $0 \in \mathring{K}$, $1 \notin K$ and $L \cap (K \cup D) = \varnothing$. Then there exists, for every $\varepsilon > 0$ and every $F \in A(L)$, a polynomial $p(z) = \sum_{k=0}^{N} a_k z^k$ which satisfies the following conditions*:

$$|a_k| < \varepsilon \qquad \textit{for} \quad k = 0, \ldots, N, \tag{1}$$

$$p(1) = 1, \tag{2}$$

$$\sum_{k=0}^{N} |a_k| < 1 + \varepsilon, \tag{3}$$

$$|p(z)| < \varepsilon \qquad \textit{for} \quad z \in (K \cup D) \backslash M(K), \tag{4}$$

$$|p(z) - F(z)| < \varepsilon \qquad \textit{for} \quad z \in L. \tag{5}$$

Proof. Define (positive!) constants ρ, δ by the equation

$$\rho = 1 - \delta/2 = \operatorname{dist}(0, K^c)$$

and consider, for $0 < \sigma < 1$, the Möbius transformations $T_\sigma(z) = \sigma/(1 + \sigma - z)$ and the homeomorphisms $h_\sigma : \mathbb{C} \to \mathbb{C}$ defined by

$$\begin{aligned} h_\sigma(z) &= (z/\rho)(1 + \sigma\rho) && \text{for} \quad |z| \leqslant \rho, \\ &= (z/|z|)(1 + \sigma |z|) && \text{for} \quad \rho < |z| < 1/(1 - \sigma), \\ &= z && \text{for} \quad |z| \geqslant 1/(1 - \sigma). \end{aligned}$$

For $|z| < 1 + \sigma$, we have

$$T_\sigma(z) = \sum_{k=0}^{\infty} c_k^{(\sigma)} z^k \qquad \text{where} \qquad c_k^{(\sigma)} = \sigma(1 + \sigma)^{-k-1}.$$

As the coefficients $c_k^{(\sigma)}$ are positive and $T_\sigma(1) = 1$, it follows that

$$\sum_{k=0}^{\infty} |c_k^{(\sigma)}| = 1. \tag{6}$$

We are now going to show that, if σ is small enough, we have

$$|c_k^{(\sigma)}| < \varepsilon/2 \qquad \text{for} \quad k = 0, \ldots, N, \tag{7}$$

$$|T_\sigma(z)| < \varepsilon/2 \qquad \text{for} \quad |z - 1| \geqslant \delta, \tag{8}$$

$$h_\sigma(L) = L, \tag{9}$$

$$K \backslash h_\sigma(K) \subset M(K). \tag{10}$$

In fact, (7) follows if $\sigma < \varepsilon/2$. If $\sigma < \min\{\delta/2, \delta\varepsilon/4\}$, then $|z - 1| \geqslant \delta$ implies $|T_\sigma(z)| = \sigma/|(z - 1) - \sigma| \leqslant \sigma/(\delta - \sigma) < (\delta\varepsilon/4)/(\delta - \delta/2)$, and so (8) follows. If $L = \varnothing$ then there is nothing to prove for (9). In the case $L \neq \varnothing$ choose $\sigma \leqslant d/(1 + d)$, where $d = \operatorname{dist}(L, \mathbb{D})$. Note that d is positive since $L \cap \bar{\mathbb{D}} = \varnothing$. With this choice for σ, $z \in L$ implies $|z| \leqslant 1 + d \geqslant 1/(1 - \sigma)$ and hence $h_\sigma(z) = z$; and therefore (9) follows.

The inclusion (10) holds if $\sigma < \delta/3$. To show this let $z \in K \backslash h_\sigma(K)$. Then z must satisfy the inequality $1 + \sigma\rho < |z| < 1/(1 - \sigma)$ because z is not an element of $h_\sigma(K) \supset h_\sigma(\mathbb{D}_\rho) = \mathbb{D}_{1 + \sigma\rho}$, and $|z| \geqslant 1/(1 - \sigma)$ would imply $z = h_\sigma(z) \in h_\sigma(K)$. A simple computation shows that $h_\sigma^{-1}(z) = z/|z| \cdot (|z| - 1)/\sigma$. Also $h_\sigma^{-1}(z) \notin K$ since $z \notin h_\sigma(K)$, and therefore we have $\operatorname{dist}(z, K^c) \leqslant |z - h_\sigma^{-1}(z)| = |1/\sigma - ((1 - \sigma)/\sigma)|z||$. As $1 + \sigma\rho < |z| < 1/(1 - \sigma)$ this is less than $1/\sigma - ((1 - \sigma)/\sigma)(1 + \sigma\rho) = \sigma + ((1 - \sigma)/2)\delta$. So if $\sigma < \delta/3$, then

$\operatorname{dist}(z, K^c) < \delta$ and also $1 < |z| < 1/(1-\sigma) < 1/(1-\delta/3) < 1+\delta$ (since $\delta = 2-2\rho < 2$), and (10) follows.

For the rest of the proof let σ be a fixed positive number such that (7), (8), (9), and (10) hold. Consider the function

$$\begin{aligned} \psi(z) &= T_\sigma(z) && \text{for} \quad z \in h_\sigma(K), \\ &= (\sigma/(z-1))(F(z)-1) && \text{for} \quad z \in L. \end{aligned}$$

Note that $1 \notin L$, that $T_\sigma(z)$ has its only pole at $1+\sigma = h_\sigma(1)$ which is not an element of $h_\sigma(K)$, and that the sets $L = h_\sigma(L)$ and $h_\sigma(K)$ are disjoint. So ψ is well defined and $\psi \in A(L \cup h_\sigma(K))$. Furthermore, as $h_\sigma\colon \mathbb{C} \to \mathbb{C}$ is a homeomorphism, $L \cup h_\sigma(K) = h_\sigma(L \cup K)$ is a compact set that does not separate the plane. Therefore, we may apply the Mergelyan approximation theorem to ψ on $L \cup h_\sigma(K)$.

So let $R = \max\{|z| : z \in L \cup h_\sigma(K)\}$, $\eta > 0$. There exists a polynomial q such that

$$|q(z) - \psi(z)| < \eta \qquad \text{for} \quad z \in L \cup h_\sigma(K). \tag{11}$$

We define a polynomial p by writing

$$p(z) = \sum_{k=0}^{N} a_k z^k = 1 + ((z-1)/\sigma)\, q(z).$$

Clearly p satisfies (2), and the theorem will be proved if we can show that, if η is small enough, the statements (1), (3), (4), and (5) also hold.

The estimate (5) follows if $\eta < \sigma\varepsilon/(R+1)$, because for $z \in L$, (11) implies

$$\begin{aligned} |p(z) - F(z)| &= |(p(z)-1) - (F(z)-1)| \\ &= (|z-1|/\sigma)\, |q(z) - \psi(z)| < ((R+1)/\sigma)\eta. \end{aligned}$$

For the other inequalities note first that we have, on $h_\sigma(K)$,

$$\begin{aligned} |p(z) - T_\sigma(z)| &= |(1 + ((z-1)/\sigma)\, q(z)) - (1 + ((z-1)/\sigma)\, T_\sigma(z))| \\ &= (|z-1|/\sigma)\, |q(z) - \psi(z)|. \end{aligned}$$

Together with (11) this yields

$$|p(z) - T_\sigma(z)| < ((R+1)/\sigma)\eta \qquad \text{for} \quad z \in h_\sigma(K). \tag{12}$$

Taking complements with respect to K in (10) we obtain $K \backslash M(K) \subset h_\sigma(K)$. Since $\mathbb{D}$ is also contained in $h_\sigma(K)$ we get $(K \cup \mathbb{D}) \backslash M(K) \subset h_\sigma(K)$. So (12) holds for all $z \in (K \cup \mathbb{D}) \backslash M(K)$. Therefore, (4) follows if we choose $\eta < \sigma\varepsilon/(2(R+1))$ and use the estimate (8) in (12).

Finally, we shall obtain (1) and (3) for small η by making use of Cauchy's integral formula

$$a_k - c_k^{(\sigma)} = \frac{1}{2\pi i} \int_{|z| = 1+\sigma\rho} \frac{p(z) - T_\sigma(z)}{z^{k+1}}\, dz \qquad \text{for} \quad k = 0, \ldots, N.$$

Estimating the integrand with (12) we get

$$|a_k| \leqslant |c_k^{(\sigma)}| + (1+\sigma\rho)^{-k}\, \eta((R+1)/\sigma) \qquad \text{for} \quad k = 0, \ldots, N. \tag{13}$$

Together with (7) this estimate implies $|a_k| < \varepsilon/2 + ((R+1)/\sigma)\eta$ ($k = 0, \ldots, N$) and therefore (1) holds if $\eta < \sigma\varepsilon/(2(R+1))$.

Also, it follows from (13) and (6) that

$$\sum_{k=0}^{N} |a_k| \leqslant \sum_{k=0}^{\infty} |c_k^{(\sigma)}| + \sum_{k=0}^{\infty} (1+\sigma\rho)^{-k}\, \eta \frac{R+1}{\sigma} = 1 + \frac{1+\sigma\rho}{\sigma^2\rho}(R+1)\eta.$$

Therefore, (3) is satisfied if $\eta < \sigma^2\rho\varepsilon/((1+\sigma\rho)(R+1))$. This completes the proof of the theorem.

In Section 3 we shall see how Theorem 1 can be applied to obtain existence theorems in summability theory. The following lemma will be useful in our applications.

LEMMA 2. *For every simply connected region G satisfying $1 \notin G$ and $\mathbb{D} \subset G$ there exists a nondecreasing sequence of compact sets $K_0 \subset K_1 \subset \cdots$ which do not separate the plane and satisfy the following conditions:*

$$0 \in \mathring{K}_n,\ 1 \notin K_n \textit{ and } K_n \subset G \textit{ for } n = 0, 1, \ldots. \tag{14}$$

To every compact subset of K of G there is an index $n_0 \in \mathbb{N}_0$ such that $K \subset K_n \backslash M(K_n)$ for $n \geqslant n_0$. (15)

To every compact subset K of $\mathbb{C} \backslash \{1\}$ there is an index $n_0 \in \mathbb{N}_0$ such that $K \subset \mathbb{C} \backslash M(K_n)$ for $n \geqslant n_0$. (16)

Proof. For $n = 0, 1, \ldots$ define K_n to be the set of all points in G the moduli of which do not exceed $n+1$ and whose distance to the complement of G is not less than $1/(n+2)$. Clearly, we have $0 \in \mathring{K}_n$ (since $\mathbb{D} \subset G$), $1 \notin K_n$ (since $K_n \subset G$ and $1 \notin G$), and $K_0 \subset K_1 \subset \cdots$. The complement of K_n can be written in the form $K_n^c = \mathbb{D}_{n+1}^c \cup U_{1/(n+2)}(G^c)$, which shows that K_n^c is connected (since G is simply connected). So K_n does not separate the plane and satisfies (14) for $n = 0, 1, \ldots$.

Let K be a compact subset of G. We may assume that $0 \in \mathring{K}$ (otherwise replace K by $K \cup \overline{\mathbb{D}}_{1/2}$). Let d be the distance between K and G^c (so

$0 < d < 1$), and let $M = \{|z| : z \in K\}$. It follows from the definition of the K_n that there is an index $n_0 \in \mathbb{N}_0$ such that

$$K_n^c \subset \mathbb{D}_{M+1}^c \cup U_{d/3}(G^c) \qquad \text{for} \quad n \geqslant n_0. \tag{17}$$

Let $n \geqslant n_0$. Since $G \supset \mathring{\mathbb{D}}$, (17) implies that $K_n \supset \{z : |z| \leqslant 1 - d/3\}$. Therefore we can write $\operatorname{dist}(0, K_n^c) = 1 - \delta/2 \geqslant 1 - d/3$ which implies $\delta \leqslant \frac{2}{3}d$. It follows from the definition of $M(K_n)$ that $M(K_n) \subset U_{2d/3}(K_n^c)$. Taking $\frac{2}{3}d$-neighbourhoods in (17) we obtain that $U_{2d/3}(K_n^c) \subset K^c$. Thus we have shown that $M(K_n) \subset K^c$. By taking complements in (17) we also find that $K \subset K_n$. Hence (15) is satisfied.

Finally, let K be a compact subset of $\mathbb{D} \backslash \{1\}$. We may again assume that $0 \in \mathring{K}$. Let $d = \operatorname{dist}(1, K)$. It follows from the definition of the K_n and the fact that $G \supset \mathring{\mathbb{D}}$ that there is an index $n_0 \in \mathbb{N}_0$ such that $K_n \supset \mathbb{D}_{1-d/2}$ for $n \geqslant n_0$. For such n, we therefore obtain that $\operatorname{dist}(0, K_n^c) \geqslant 1 - d/2$ implying $\mathbb{D} \backslash M(K_n) \supset \mathbb{D} \backslash U_d(1) \supset K$. Thus we have proved the remaining part (16) of the lemma.

3. Regular Summability Methods Which Sum the Geometric Series to Prescribed Values outside the Unit Circle

With the results in Section 2 we are now able to construct regular matrices which sum the geometric series to prescribed functions on sets larger than the unit circle. It will be convenient to use the abbreviations

$$\tau_n^A(z) = \sum_{k=0}^{\infty} a_{n,k} z^k, \qquad \sigma_n^A(z) = \sum_{k=0}^{\infty} a_{n,k}(1 + \cdots + z^k)$$

for $n \geqslant 0$, $z \in \mathbb{C}$, and row-finite $A = (a_{n,k})_{n,k=0}^{\infty}$. If, in addition, $\tau_n^A(1) = 1$ then it is easy to see that

$$\sigma_n^A(z) = (1/(1-z))(1 - z\tau_n^A(z)) \qquad \text{for} \quad z \neq 1. \tag{18}$$

As a simple consequence of Theorem 1 we obtain

THEOREM 3. *Let G be a simply connected region the complement of which is a simple path Γ joining 1 and ∞ and intersecting the closed unit disc only at 1. Then there exists a matrix $A \in \mathscr{R}$ such that*

(a) $\lim_{n \to \infty} \tau_n^A(z) = 0$ *and* $\lim_{n \to \infty} \sigma_n^A(z) = 1/(1-z)$ *for all* $z \in \mathbb{C} \backslash \{1\}$,

(b) *the sequences (σ_n^A), (τ_n^A) both converge compactly on each one of the sets G and $\Gamma \backslash \{1\}$.*

In the case where G is the Mittag–Leffler star $\mathbb{C} \backslash [1, \infty)$ explicit methods

(Lindelöf, Mittag–Leffler, LeRoy—see, e.g., [3]) are known to sum the geometric series to $1/(1-z)$ on G. Part (a) of Theorem 3 says that we can do better than that. Of course, the problem of finding explicit matrices remains open. (For explicit methods satisfying (a) and (b) see [14] or [15].)

Proof of Theorem 3. We may assume that the parametrisation of Γ is such that $\Gamma(1)=1$ and $\lim_{t\to\infty}\Gamma(t)=\infty$. Define $L_n=\Gamma([1+1/(n+1), n+2])$ for $n\geqslant 0$ and let K_n be compact sets like in Lemma 2. Theorem 1 with $K=K_n$, $L=L_n$, $\varepsilon=1/(n+1)$, and $F(z)=0$ guarantees the existence of polynomials $p_n(z)=\sum_{k=0}^{\infty} a_{n,k}z^k$ $(n=0,1,...)$ satisfying

$$|a_{n,k}|<1/(n+1) \qquad \text{for} \quad k=0,1,.... \tag{19}$$

$$\sum_{k=0}^{\infty} a_{n,k}=1, \tag{20}$$

$$\sum_{k=0}^{\infty} |a_{n,k}|<1+1/(n+1), \tag{21}$$

$$|p_n(z)|<1/(n+1) \qquad \text{for} \quad z\in(K_n\cup\mathbb{D})\setminus M(K_n) \text{ and for } z\in L_n. \tag{22}$$

Consider the matrix $A=(a_{n,k})$. Clearly (19), (20), (21) imply the regularity of A and we even have $A\in\mathbb{R}$. The transforms τ_n^A are equal to the polynomials p_n. Since (18) also holds it suffices to prove that $\lim_{n\to\infty}p_n(z)=0$ uniformly on every compact subset of G and on every compact subset of $\Gamma\setminus\{1\}$. But every such set is, for sufficiently large n, either contained in L_n or—because of (15)—a subset of $(K_n\cup\mathbb{D})\setminus M(K_n)$. In either case (22) implies the statements (a) and (b).

In another application of the approximation Theorem 1 we are now going to improve a result by Luh (Theorem 5.1 in [9]) in which he constructs methods A which satisfy conditions (23) through (27) of the following theorem, but does not obtain regularity. For further applications of Theorem 1 see, for example, [1] or [15].

THEOREM 4. *Suppose that G_0 is a simply connected region containing the open unit disc but not the point* 1. *Also suppose that at most countably many*

simply connected regions $G_1, G_2, ...,$
Jordan arcs $J_1, J_2, ...,$

and

point sets $\{z_1\}, \{z_2\}, ...$

are given, and that these sets are disjoint and contain no point of $\mathbb{D}\cup G_0$.

Furthermore, let $f_0(z) = 1/(1-z)$ *and let, for* $\nu \geqslant 1$,

$f_\nu(z)$ *be holomorphic on* G_ν,
$g_\nu(z)$ *be continuous on* J_ν,
ω_ν *be any complex number.*

Then there exists a matrix $A \in \mathscr{R}$ *which satisfies the following conditions*:

$$\lim_{n\to\infty} \sigma_n^A(z) = f_\nu(z) \qquad \text{compactly on } G_\nu, \text{for each } \nu \geqslant 0, \tag{23}$$

$$\lim_{n\to\infty} \sigma_n^A(z) = 1/(1-z) \qquad \text{compactly on } \mathbb{D}\backslash\{1\}, \tag{24}$$

$$\lim_{n\to\infty} \sigma_n^A(z) = g_\nu(z) \qquad \text{uniformly on } J_\nu, \text{for each } \nu \geqslant 1, \tag{25}$$

$$\lim_{n\to\infty} \sigma_n^A(z) = \omega_\nu \qquad \text{for each } \nu \geqslant 1, \tag{26}$$

$$\text{the sequence } (\sigma_n^A(z)) \text{ diverges for}$$
$$z \notin \mathbb{D} \cup \bigcup_{\nu\geqslant 0} G_\nu \cup \overline{\bigcup_{\nu\geqslant 1} J_\nu} \cup \overline{\{z_1,\ldots\}}. \tag{27}$$

Proof. For notational ease, define $G_\nu = \varnothing$ resp. $J_\nu = \varnothing$ resp. $\{z_\nu\} = \varnothing$ wherever the index ν exceeds the number of sets given. Let $K_n = K_{n,0}$ be compact sets like in Lemma 2 with $G = G_0$. Every G_ν $(\nu \geqslant 1)$ can be exhausted by a sequence of compact sets $K_{n,\nu}$ not separating the plane, so that we have, for every $\nu \geqslant 1$,

$$\begin{aligned}&\text{to every compact subset } K \text{ of } G_\nu \text{ there is an index}\\ &n_0 \geqslant 0 \text{ such that } K \subset K_{n,\nu} \subset G_\nu \text{ for } n \geqslant n_0.\end{aligned} \tag{28}$$

For $n = 0, 1,\ldots$, let d_n denote the (positive!) distance between the sets

$$H_n = \left(\bigcup_{\nu=0}^{n} G_\nu\right)^c \qquad \text{and} \qquad \bigcup_{\nu=0}^{n} K_{n,\nu}.$$

Each of the compact sets

$$E_n = (\mathbb{D}_n \cap H_n)\backslash U_{d_n}\left(\mathbb{D} \cup \bigcup_{\nu=1}^{n} (J_\nu \cup \{z_\nu\})\right)$$

can covered by finitely many compact discs $D_{n,2},\ldots, D_{n,r_n}$ with radii $d_n/2$ and centers in E_n. We define $D_{n,1} = \varnothing$ and

$$L_{n,\rho} = D_{n,\rho} \cup \bigcup_{\nu=1}^{n} (K_{n,\nu} \cup J_\nu \cup \{z_\nu\}) \qquad \text{for} \quad n = 0, 1,\ldots, 1 \leqslant \rho \leqslant r_n.$$

It is not hard to verify that, for every fixed pair (n, ρ), the sets

$$D_{n,\rho}, \qquad K_{n,1}, \dots, K_{n,n}, \qquad J_1, \dots, J_n, \qquad \{z_1\}, \dots, \{z_n\}$$

are disjoint and do not separate the plane. Hence $L_{n,\rho}$ does not separate the plane; and it is not hard to show that $L_{n,\rho}$ does not intersect $\bar{D} \cup K_n$. Therefore, we may apply Theorem 1 to $K = K_n$, $L = L_{n,\rho}$, $\varepsilon = 1/(n+1)$, and $F(z) = F_{n,\rho}(z)$, where

$$\begin{aligned} F_{n,\rho}(z) &= 1/z - ((1-z)/z) f_\nu(z) && \text{if} \quad z \in G_\nu,\ 1 \leqslant \nu \leqslant n, \\ &= 1/z - ((1-z)/z)\, g_\nu(z) && \text{if} \quad z \in J_\nu,\ 1 \leqslant \nu \leqslant n, \\ &= 1/z_\nu - (1 - z_\nu) w_\nu / z_\nu && \text{if} \quad z = z_\nu,\ 1 \leqslant \nu \leqslant n, \\ &= n && \text{if} \quad z \in D_{n,\rho}. \end{aligned}$$

Thus, there exist polynomials $p_{n,\rho}(z) = \sum_{k=0}^{\infty} a_k^{(n,\rho)} z^k$ satisfying

$$|a_k^{(n,\rho)}| < 1/(n+1) \qquad \text{for} \quad k = 0, 1, \dots, \tag{29}$$

$$\sum_{k=0}^{\infty} a_k^{(n,\rho)} = 1, \tag{30}$$

$$\sum_{k=0}^{\infty} |a_k^{(n,\rho)}| < 1 + 1/(n+1), \tag{31}$$

$$|p_{n,\rho}(z)| < 1/(n+1) \qquad \text{for} \quad z \in (K_n \cup \bar{D}) \backslash M(K_n), \tag{32}$$

$$|p_{n,\rho}(z) - F_{n,\rho}(z)| < 1/(n+1) \qquad \text{if} \quad z \in L_{n,\rho}. \tag{33}$$

Now consider the matrix $A = (a_{m,k})$, where

$$a_{m,k} = a_k^{(n,\rho)} \qquad \text{for} \quad k = 0, 1, \dots, m = \sum_{0 \leqslant \nu < n} r_\nu + \rho - 1 \quad (n \geqslant 0,\ 1 \leqslant \rho \leqslant r_n).$$

Plainly, (29) through (31) imply that $A \in \mathscr{R}$. Furthermore, we have

$$\tau_m^A(z) = p_{n,\rho}(z) \qquad \text{if} \quad \sum_{0 \leqslant \nu < n} r_\nu \leqslant m < \sum_{0 \leqslant \nu \leqslant n} r_\nu. \tag{34}$$

If K is a compact subset of G_ν $(\nu \geqslant 1)$ then (28), (33), and the fact that $K_{n,\nu} \subset L_{n,\rho}$ imply that $\lim_{m \to \infty} \tau_m^A(z) = 1/z - ((1-z)/z) f_\nu(z)$ uniformly on K. Similarly, if K is a compact subset of G_0 or a compact subset of $\bar{D} \backslash \{1\}$ then (15) resp. (16) together with (32) imply that $\lim_{m \to \infty} \tau_m^A(z) = 0$ uniformly on K. Inserting these limits in (18) the assertions (23) and (24) follow. The statements (25) and (26) are proved analogously, after observing that $J_n \cup \{z_n\} \subset L_{n,\rho}$ $(n = 0, \dots, 1 \leqslant \rho \leqslant r_n)$.

Finally, suppose that z is as in (27). Then z has a positive distance d to the set $\mathbb{D} \cup \overline{\bigcup_{v \geqslant 1} J_v} \cup \overline{\{z_1, ...\}}$ and also $z \in H_n$ for every $n \in \mathbb{N}_0$. Because of the identity (18), it suffices to show that, for this fixed z, the sequence $(\tau_m^A(z))$ is unbounded. Since $\lim_{n \to \infty} d_n = 0$, it follows that z is an element of E_n, for sufficiently large n. Hence there is a number $n_0 \in \mathbb{N}$ such that, for every $n \geqslant n_0$, z lies in at least one of the discs $D_{n,2}, ..., D_{n,r_n}$, say,

$$z \in D_{n,\rho_n} \qquad \text{for} \quad n \geqslant n_0.$$

The estimate (33) and the definition of F_{n,ρ_n} show that

$$|p_{n,\rho_n}(z)| > n - 1/(n+1) \qquad \text{for} \quad n \geqslant n_0.$$

So the sequence $(p_{n,\rho_n}(z))$ is unbounded and, because of (34), it is also a subsequence of $(\tau_m^A(z))$. Thus, the sequences $(\tau_m^A(z))$ and $(\sigma_m^A(z))$ both diverge.

This completes the proof of the theorem.

References

1. K. Faulstich, W. Luh, and L. Tomm, Universelle Approximation durch Riesz-Transformierte der geometrischen Reihe, *Manuscripta Math.* **36** (1981), 309–321.
2. W. Gawronski and R. Trautner, Verschärfung eines Satzes von Borel–Okada über Summierbarkeit von Potenzreihen, *Period. Math. Hungar.* **7**(3–4) (1976), 201–211.
3. G. H. Hardy, "Divergent Series," Oxford Univ. Press (Clarendon), Oxford, 1963.
4. R. B. Israpilov, The summation of power series at isolated points outside the circle of convergence (Russian), *Sakharth. SSR Mecn. Akad. Moambe* **54** (1969), 15–16.
5. R. B. Israpilov, Sets of summability of power series by linear methods (Russian), *Vestnik Moskov. Univ. Ser. I Mat. Meh.* **24**(3) (1969), 22–29.
6. R. B. Israpilov, The summability of power series outside the circle of convergence (Russian), *Isv. Vysš. Učebn. Zaved. Matematika* **93**(2) (1970), 19–26.
7. A. Jakomovski, Analytic continuation and summability of power series, *Michigan Math. J.* **11** (1964), 353–356.
8. W. Luh, Approximation analytischer Funktionen durch überkonvergente Potenzreihen und deren Matrix-Transformierten, *Mitt. Math. Sem. Giessen* **88** (1970).
9. W. Luh, Über die Summierbarkeit der geometrischen Reihe, *Mitt. Math. Sem. Giessen* **113** (1974).
10. W. Luh and R. Trautner, Summierbarkeit der geometrischen Reihe auf vorgeschriebenen Mengen, *Manuscripta Math.* **18** (1976), 317–326.
11. W. Rudin, "Real and Complex Analysis," McGraw-Hill, New York, 1974.
12. D. C. Russell, Summability of power series on continuous arcs outside the circle of convergence, *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **45** (1959), 1006–1030.
13. S. E. Tolba, On the summability of Taylor series at isolated points outside the circle of convergence, *Nederl. Akad. Wetensch. Proc. Ser. A.* **55** (1952), 380–387.
14. L. Tomm, A regular summability method which sums the geometric series to its proper value in the whole complex plane, *Canad. Math. Bull.* **26** (2) (1983), 179–188.
15. L. Tomm, "Über die Summierbarkeit der geometrischen Reihe mit regulären Verfahren," Dissertation, Ulm, West Germany, 1979.
16. P. Vermes, On γ-matrices and their application to the binomial series, *Proc. Edinburgh Math. Soc.* **8**(2) (1947), 1–13.

17. P. VERMES, Conservative series to series transformation matrices. *Acta Sci. Math. (Szeged)* **14** (1951), 23–28.
18. P. VERMES, Convolution of summability methods, *J. Analyse Math.* **2** (1952), 160–177.
19. P. VERMES, Summability of power series in simply or multiply connected domains. *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **44** (1958), 188–198.
20. P. VERMES, Summability of power series at unbounded sets of isolated points. *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **44** (1958), 830–838.